Product of 2-by-2 matrices and cutpoints of random walks

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Part I:
Products of Nonnegative 2-by-2 Matrices

Product of 2×2 Matrices

Let $M_k := \begin{pmatrix} a_k & b_k \\ d_k & \theta_k \end{pmatrix}, k \geq 1$ be nonnegative matrices.

Consider the products $M_{k+1} \cdots M_{k+n}$.

In ergodic theory of product of nonnegative matrices, it is shown that $\frac{\mathbf{e}_1 M_{k+1} \cdots M_{k+n} \mathbf{e}_j}{\mathbf{e}_2 M_{k+1} \cdots M_{k+n} \mathbf{e}_j} \to c, j = 1, 2, n \to \infty$, see Seneta (1981).

Further Question: $\mathbf{e}_i M_{k+1} \cdots M_{k+n} \mathbf{e}_j \sim ?$

Motivations:

- the escape prob. of (1,2) and (2,1) random walks are functions of $\mathbf{e}_i M_{k+1} \cdots M_{k+n} \mathbf{e}_j$;
- The asymptotics of 2-type branching processes rely on the limit behaviours of $\mathbf{e}_i M_{k+1} \cdots M_{k+n} \mathbf{e}_j$.

Seneta E. Non-negative matrices and Markov chain. 2nd Ed. Newyork: Springer Science & Business Media, LLC; 1981.

Some continued fractions related to $M_k, k \geq 1$

For $n \geq 1$, let

$$\beta_n = \frac{b_{n+1}}{b_n(b_{n+1}d_{n+1} - a_{n+1}\theta_{n+1})}, \alpha_n = \frac{a_n b_{n+1} + b_n \theta_{n+1}}{b_n(b_{n+1}d_{n+1} - a_{n+1}\theta_{n+1})}$$

and set

$$\xi_n := \frac{\beta_n}{\alpha_n} + \frac{\beta_{n+1}}{\alpha_{n+1}} + \frac{\beta_{n+2}}{\alpha_{n+2}} + \cdots.$$

Let $M := \begin{pmatrix} a & b \\ d & \theta \end{pmatrix}$ be a nonnegative matrix and let

$$\varrho := \frac{a + \theta + \sqrt{(a+\theta)^2 + 4(bd - a\theta)}}{2},$$

$$\varrho_1 := \frac{a + \theta - \sqrt{(a+\theta)^2 + 4(bd - a\theta)}}{2}$$

be the eigenvalues of M.

Theorem 1

Suppose that $\lim_{k\to\infty} M_k = M$, $a+\theta \neq 0$, $b \neq 0$ and $bd \neq a\theta$. Then $\exists k_0 > 0$ such that for $k \geq k_0$ and $i, j \in \{1, 2\}$, we have

$$\frac{\mathbf{e}_i M_{k+1} \cdots M_{k+n} \mathbf{e}_j^t}{\xi_{k+1}^{-1} \cdots \xi_{k+n}^{-1}} \to \psi(i, j, k),$$

uniformly in k as $n \to \infty$; furthermore, if we assume further $\varrho \geq 1$, then for $k \geq k_0$ and $i = 1$, $j \in \{1, 2\}$, with the above $\psi(i, j, k)$, we have

$$\frac{\sum_{s=1}^{n+1} \mathbf{e}_i M_{k+s} \cdots M_{k+n} \mathbf{e}_j^t}{\sum_{s=1}^{n+1} \xi_{k+s}^{-1} \cdots \xi_{k+n}^{-1}} \to \psi(i, j, k),$$

uniformly in k as $n \to \infty$,

Wang, H.-M.: Asymptotics of entries of products of nonnegatives 2-by-2 matrices, *arXiv*: 2111.10232 (2022)

where

$$\psi(1,1,k) = \frac{\varrho - \theta}{\varrho - \varrho_1}, \psi(1,2,k) = \frac{b}{\varrho - \varrho_1},$$
$$\psi(2,1,k) = \frac{\varrho}{\varrho - \varrho_1}\left(\frac{\theta_{k+1}}{b_{k+1}} - \xi_{k+1}\frac{\det(M_{k+1})}{b_{k+1}}\right),$$
$$\psi(2,2,k) = \frac{b}{\varrho - \varrho_1}\left(\frac{\theta_{k+1}}{b_{k+1}} - \xi_{k+1}\frac{\det(M_{k+1})}{b_{k+1}}\right).$$

Remark

- ◇ According to the above theorem, entries of the product of matrices can be evaluated by the product of continued fractions, which are some **numbers**.
- ◇ If $\varrho(M_n)$ is the spectral radius of M_n, it is easy to show that

$$\varrho(M_n) \sim \xi_n, n \to \infty.$$

Lyapunov Exponent of product of 2-by-2 i.i.d. random matrices

Proposition

Suppose $M_n, n \geq 1$ are i.i.d.(or ergodic), $P(\det(M_1) > 0) = 1$ and $E \log^+(\max\{a_1, b_1, d_1, \theta_1\}) < \infty$. Then for $i, j \in \{1, 2\}$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbf{e}_i M_1 \cdots M_n \mathbf{e}_j = E(\log \xi_1^{-1}), \ P\text{-a.s.}.$$

The proposition is proved based on the following observations:

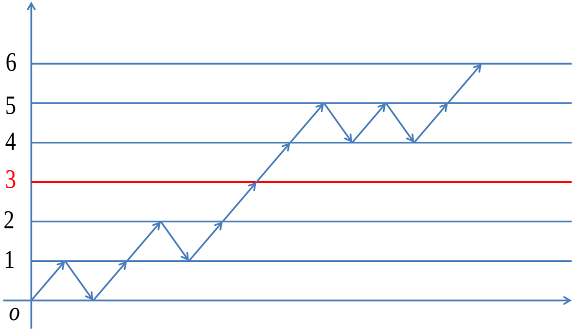
$$\xi_{k,n}^{-1} \cdots \xi_{n,n}^{-1} = \mathbf{e}_1 A_k \cdots A_n \mathbf{e}_1^t,$$

$$\xi_k \cdots \xi_n \leq \xi_{k,n} \cdots \xi_{n,n} \leq \xi_k \cdots \xi_{n-1} \xi_{n,n}, 1 \leq k \leq n.$$

Part II:

Cut-point Problem of Random Walks

Nearest-Neighbor Random Walk



Occupation time of the site 3 is **1**, it 'cuts' the path of the RW into two disjoint parts, the 'future' and the 'past', so the site 3 is a cutpoint.

- ♡ Intuitively, the **faster** the walk runs, the **more cutpoints** it has.
- ♡ Is it possible that the walk is **transient**, but not fast enough, so that there are only **finitely many cutpoints?**

Suppose that $\{X_n\}$ is a random walk on $\mathbb{Z}_+$ with trans. prob.

$$P(X_{n+1} = i+1 | X_n = i) = p_i \in (0,1),$$
$$P(X_{n+1} = i-1 | X_n = i) = q_i = 1 - p_i, n \geq 0, i \geq 1;$$
$$P(X_{n+1} = 1 | X_n = 0) = 1, n \geq 0.$$

- ♢ For the above model, James, Lyons, Peres (2008) give a example which shows that the walk is **transient** but has only **finitely** many cutpoints.
- ♢ Csáki, Földes, Révész (2010) give a **criterion** for the finiteness of the number of cutpoints.

Csáki, E., Földes, A., Révész, P.: On the number of cutpoints of transient nearest neighbor random walk on the line. *J. Theor. Probab.* **23**, 624-638 (2010)

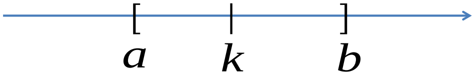
James, N., Lyons, R., Peres, Y.: A transient Markov chain with finitely many cutpoints. *In: IMS Collections Probability and Statistics: Essays in Honor of David A. Freedman*, **2**, 24-29, Institute of Mathematical Statistics (2008)

Set $\rho_i = \frac{q_i}{p_i}, i \geq 1$, and $D(n) = 1 + \sum_{j=1}^{\infty} \rho_{n+1} \cdots \rho_{n+j}, n \geq 0$.

Theorem(Csáki, Földes, Révész)

Suppose $0 \leq p_i < 1/2, i \geq 1$.

- if
$$\sum_{n=1}^{\infty} \frac{1}{D(n) \log n} < \infty,$$
then almost surely, $\{X_n\}$ has at most **finitely** many cut-points;
- If $\exists \delta > 0$ such that $D(n) \leq \delta n \log n$ for n large enough and
$$\sum_{n=1}^{\infty} \frac{1}{D(n) \log n} = \infty,$$
then almost surely, $\{X_n\}$ has **infinitely** many cutpoints.



$$P_k(a,b,-) = P(X \text{ hits } a \text{ before it hits } b | X_0 = k).$$

Lemma

for $0 \leq a \leq k \leq b$, we have

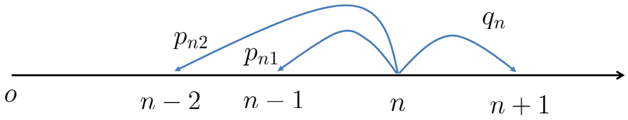
$$P_k(a,b,-) = \frac{\sum_{j=k}^{b-1} \rho_{a+1} \cdots \rho_j}{1 + \sum_{j=a+1}^{b-1} \rho_{a+1} \cdots \rho_j}.$$

- ◇ The proof of the above theorem is based on some delicate analysis of the escape probabilities. The formulae of such escape probabilities are very simple and the value of ρ_i is known in advance.

We aim to discuss the asymptotics of the number cutpoints of some **non-nearest neighbour** random walk. Now, we can deal with such problem for **(1,2)** and **(2,1)** random walks.

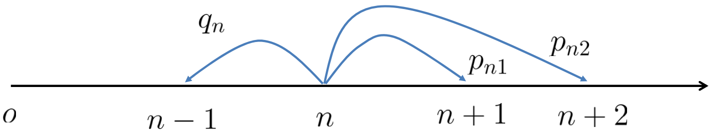
♢ **(2,1) random walk** $X = \{X_k\}_{k\geq 0}$:

$$P(X_{k+1} = 1|X_k = 0) = P(X_{k+1} = 2|X_k = 1) = 1,$$
$$P(X_{k+1} = n+1|X_k = n) = q_n,$$
$$P(X_{k+1} = n-1|X_k = n) = p_{n1},$$
$$P(X_{k+1} = n-2|X_k = n) = p_{n2}, n \geq 2, k \geq 0.$$



$\diamondsuit$ **(1,2) random walk** $Y = \{Y_k\}_{k\geq 0}$:

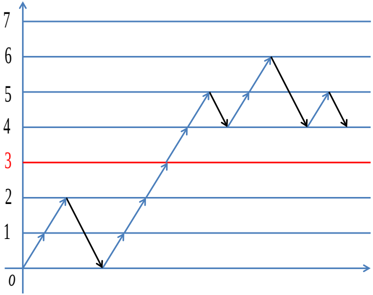
$$P(Y_{k+1} = 0|Y_k = 1) = P(Y_{k+1} = 2|Y_k = 0) = 1,$$
$$P(Y_{k+1} = n-1|Y_k = n) = q_n,$$
$$P(Y_{k+1} = n+1|Y_k = n) = p_{n1},$$
$$P(Y_{k+1} = n+2|Y_k = n) = p_{n2}, n \geq 2, k \geq 0.$$



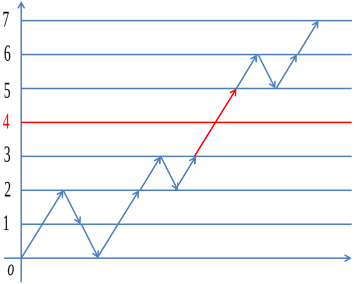
Cutpoints of (1,2) and (2,1) Random Walks

Definition

- For **(2,1)** random walk X, if $\#\{n \geq 0 : X_n = k\} = 1$, we say k is a cutpoint of X;
- for **(1,2)** random walk Y, if $\#\{n \geq 0 : X_n = k\} = 0$, we say k is a cutpoint(or Skipped Point) of Y. .



(2,1) random walk



(1,2) random walk

Escape Probabilities

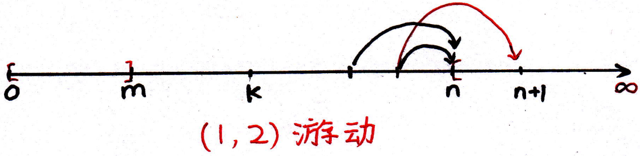
- ◇ To study the cutpoint of (1,2) and (2,1) random walks, the difficulty is that the escape probabilities are functions of products of 2-by-2 matrices, whose entries are hard to evaluate. We have treated these evaluations in the first part.

Consider **(1,2) random walk** Y. For $2 \le m \le k \le n+1$ and $j \in \{n, n+1\}$, set

$$Q_k^j(m,n) = P(Y \text{ hits } [n,\infty) \text{ at } j \text{ before it hits } [0,m] \big| Y_0 = k),$$

and set

$$Q_k(m,n,+) := Q_k^n(m,n) + Q_k^{n+1}(m,n).$$



Let $A_k := \begin{pmatrix} a_k & b_k \\ 1 & 0 \end{pmatrix}$ where $a_k := \frac{p_{k1}+p_{k2}}{q_k}, b_k = \frac{p_{k2}}{q_k}$.

It is easy to compute the **spectral radius**(top eigenvalue) of A_k,
$\varrho_k = \frac{a_k+\sqrt{a_k^2+4b_k}}{2}$

Lemma

Consider **(1,2) random walk** Y. For $1 \le m < k < n$, we have

$$Q_k^n(m,n) = \sum_{s=m+1}^{k} \mathbf{e}_1 A_s \cdots A_{n-1} \left(\frac{1+\sum_{s=m+1}^{n-1} \mathbf{e}_1 A_s \cdots A_{n-1} \mathbf{e}_2^t}{1+\sum_{s=m+1}^{n-1} \mathbf{e}_1 A_s \cdots A_{n-1} \mathbf{e}_1^t} \mathbf{e}_1^t - \mathbf{e}_2^t \right),$$

$$Q_k^{n+1}(m,n) = \sum_{s=m+1}^{k} \mathbf{e}_1 A_s \cdots A_{n-1} \left(\mathbf{e}_2^t - \frac{\sum_{s=m+1}^{n-1} \mathbf{e}_1 A_s \cdots A_{n-1} \mathbf{e}_2^t}{1+\sum_{s=m+1}^{n-1} \mathbf{e}_1 A_s \cdots A_{n-1} \mathbf{e}_1^t} \mathbf{e}_1^t \right),$$

$$Q_k(m,n,+) = \frac{\sum_{s=m+1}^{k} \mathbf{e}_1 A_s \cdots A_{n-1} \mathbf{e}_1^t}{1+\sum_{s=m+1}^{n-1} \mathbf{e}_1 A_s \cdots A_{n-1} \mathbf{e}_1^t}.$$

Escape probabilities for (2,1) random walk X

For $0 < m - 1 \leq k \leq n$, set

$$P_k(m, n, -) = P(X \text{ hits } [0, m] \text{ before it hits } [n, \infty) \big| X_0 = k).$$

Lemma

Consider **(2,1) random walk** X. For $1 \leq m < k < n$, we have

$$P_k(m, n, -) = \frac{\sum_{s=k}^{n-1} \mathbf{e}_1 A_s \cdots A_{m+1} \mathbf{e}_1^t}{1 + \sum_{s=m+1}^{n-1} \mathbf{e}_1 A_s \cdots A_{m+1} \mathbf{e}_1^t}.$$

Conditions

We now introduce the follow condition:

(C) *Assume* $a_k \to a > 0$, $b_k \to b > 0$.

◇ Under condition (C), $A_k \to A := \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$.

The eigenvalues of A are

$$\varrho = \frac{1}{2}\Big(a + \sqrt{a^2 + 4b}\Big), \sigma = \frac{1}{2}\Big(a - \sqrt{a^2 + 4b}\Big).$$

It is easy to see that $-1 < \sigma < 0$ and $|\sigma| < \varrho$.

◇ Under Condition (C),

$$\text{if } a + b = 1, \text{ then } \varrho_k \to \varrho = 1, p_{k1} + 2p_{k2} - q_k \to 0.$$

In this case, we say the random walk is **near recurrent** or **near critical**.

Asymptotics of hitting probabilities and escape probabilities

Consider **(1,2) random walk** Y. For $k \geq 0$, write $L_k = \{2k, 2k+1\}$. For $k \geq 1$, set

$$T_k = \inf\{n \geq 0 : Y_n \in L_k\},$$

$$h_k(1) = P(Y_{T_k} = 2k),$$

$$h_k(2) = P(Y_{T_k} = 2k+1), k \geq 1;$$

$$\eta_{k,m}(1) = P(Y \text{ hits } [m+1, \infty) \text{ at } m+1 | Y_0 = k),$$

$$\eta_{k,m}(2) = P(Y \text{ hits } [m+1, \infty) \text{ at } m+2 | Y_0 = k), m \geq k \geq 1.$$

Proposition

Suppose that (C) holds. Then

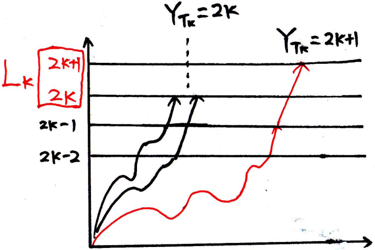
$$\lim_{n\to\infty} \eta_{n,n}(2) = -\sigma, \lim_{n\to\infty} h_n(2) = -\frac{\sigma}{1-\sigma};$$

For $k \geq 1$, set $\hat{a} := a + \frac{(1-\varrho)(\varrho-\sigma)}{\varrho}$. Then as $n \to \infty$, we have

$$\frac{Q_{k+1}^{k+n}(k, k+n)}{Q_{k+1}^{k+n+1}(k, k+n)} \to -\sigma^{-1},$$

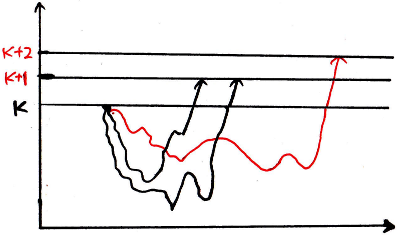
$$Q_{k+n-1}^{k+n+1}(k, k+n) \to \tau := \begin{cases} -\sigma, & \text{if } \varrho \geq 1, \\ \frac{\sqrt{\hat{a}^2+4b}-\hat{a}}{2}, & \text{if } \varrho < 1, \end{cases}$$

where the convergence is **uniform** in $k \geq 1$.



从0出发，可能在2K或2K+1处击中 $L_K=\{2K, 2K+1\}$.

$$h_K(2) \longrightarrow -\frac{\sigma}{1-\sigma}$$



从K出发，Y可能从K+1或K+2处击中 $[K+1, \infty)$

$$\eta_{K,K}(2) \longrightarrow -\sigma$$

I had considered a special (1,2) random walk, whose negative jumps are always with size 1 and **positive jumps are always with size 2**. I gave a criterion for the finiteness of cutpoints of this special model. But unfortunately, I could not get an almost-sure result for the case there are infinitely many skipped points, because I could not prove the above proposition at that period. See the paper below.

Wang, H.-M.: On the number of points skipped by a transient (1,2) random walk on the lattice of the positive half line. *Markov Processes Relat. Fields.* **25**,125-148 (2019)

Next we give criteria for the finiteness of cutpoints of (1,2) and (2,1) random walks.

Recall that

$$A_k := \begin{pmatrix} a_k & b_k \\ 1 & 0 \end{pmatrix} \text{ where } a_k := \frac{p_{k1} + p_{k2}}{q_k}, b_k = \frac{p_{k2}}{q_k},$$

$$\varrho_k = \frac{a_k + \sqrt{a_k^2 + 4b_k}}{2}$$

is the spectral radius of A_k. Define

$$D_X(n) = 1 + \sum_{j=n+1}^{\infty} \prod_{i=n+1}^{j} \varrho_i, \quad D_Y(n) = 1 + \sum_{j=n+1}^{\infty} \prod_{i=n+1}^{j} \varrho_i^{-1}.$$

X is a (2,1) RW. For X, a point with occupation time **1** is called a cutpoint.

Y is a (1,2) RW. For Y, a point with occupation time **0** is called a cutpoint(or skipped point).

Criteria for the finiteness of the number of cutpoints

Theorem 2

Consider random walk $Z \in \{X, Y\}$. Suppose that Condition (C) holds, $a+b=1$ and $\exists N_0 > 0$ such that ϱ_k is increasing in $k \geq N_0$ when $Z = X$, and decreasing in $k \geq N_0$ when $Z = Y$.

- If

$$\sum_{n=2}^{\infty} \frac{1}{D_Z(n) \log n} < \infty,$$

  then almost surely, Z has at most finitely many cutpoints.

- If there exists some $\delta > 0$ such that $D_Z(n) \leq \delta n \log n$ for n large enough and

$$\sum_{n=2}^{\infty} \frac{1}{D_Z(n) \log n} = \infty,$$

  then almost surely, Z has infinitely many cutpoints.

Fix $\beta \geq 0$ and set

$$r_n = \begin{cases} \frac{1}{3}\left(\frac{1}{n} + \frac{1}{n(\log\log n)^\beta}\right), & \text{if } n \geq 4, \\ r_4, & \text{if } n = 2, 3. \end{cases}$$

Corollary

Assume that Condition (C) holds.

(i) If $\varrho_k = 1 - 3r_k + \mathrm{O}(r_k^2)$, then almost surely ,

$\beta > 1 \Rightarrow X$ has at most finitely many cutpoints;

$\beta \leq 1 \Rightarrow X$ has infinitely many cutpoints.

(ii) If $\varrho_k = 1 + 3r_k + \mathrm{O}(r_k^2)$, then almost surely,

$\beta > 1 \Rightarrow Y$ has at most finitely many cutpoints;

$\beta \leq 1 \Rightarrow Y$ has infinitely many cutpoints.

Proof. If $\varrho_k = 1 \pm 3r_k + \mathrm{O}(r_k^2)$ as $k \to \infty$, then $\exists N_0 > 0$ such that ϱ_k is decreasing(increasing) in $k \geq N_0$. Furthermore, with $0 < c_1 < c_2 < \infty$ some proper constants, if $\varrho_k = 1 - 3r_k + \mathrm{O}(r_k^2)$ as $k \to \infty$, it can be shown that for n large enough we have

$$c_1 n(\log\log n)^\beta \leq D_X(n) \leq c_2 n(\log\log n)^\beta;$$

otherwise, if $\varrho_k = 1 + 3r_k + \mathrm{O}(r_k^2)$ as $k \to \infty$, for n large enough we have

$$c_1 n(\log\log n)^\beta \leq D_Y(n) \leq c_2 n(\log\log n)^\beta.$$

Therefore the corollary is a direct consequence of the above theorem. $\square$

When there are **infinitely** many cutpoints, one ask naturally how many cutpoints there are in $[2, n]$. The theorem below shows that the number of cutpoints in $[2, n]$ is approximately $\frac{c \log n}{(\log\log n)^{\beta}}$.

Theorem 3

Consider the chain $Z \in \{X, Y\}$. For $n \geq 2$, set

$$S_n = \#\{k \in [2, n] : k \text{ is a cutpoint of } Z\}.$$

Suppose Condition (C) holds. Assume further $\varrho_k = 1 - 3r_k + \mathrm{O}(r_k^2)$ as $k \to \infty$ if $Z = X$ and $\varrho_k = 1 + 3r_k + \mathrm{O}(r_k^2)$ as $k \to \infty$ if $Z = Y$ respectively. If $0 \leq \beta \leq 1$, then with proper constants $0 < c_3 < c_4 < \infty$, we have

$$c_3 \leq \liminf_{n\to\infty} \frac{ES_n}{\log n(\log\log n)^{-\beta}} \leq \limsup_{n\to\infty} \frac{ES_n}{\log n(\log\log n)^{-\beta}} \leq c_4,$$

and for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{S_n}{(\log n)^{1+\varepsilon}(\log\log n)^{-\beta}} = 0, \ a.s.. \tag{1}$$

Criteria for the finiteness of the number of cutpoints

Theorem 2

Consider random walk $Z \in \{X, Y\}$. Suppose that Condition (C) holds, $a+b=1$ and $\exists N_0 > 0$ such that ϱ_k is increasing in $k \geq N_0$ when $Z = X$, and decreasing in $k \geq N_0$ when $Z = Y$.

- If

$$\sum_{n=2}^{\infty} \frac{1}{D_Z(n)\log n} < \infty,$$

  then almost surely, Z has at most finitely many cutpoints.

- If there exists some $\delta > 0$ such that $D_Z(n) \leq \delta n \log n$ for n large enough and

$$\sum_{n=2}^{\infty} \frac{1}{D_Z(n)\log n} = \infty,$$

  then almost surely, Z has infinitely many cutpoints.

Lemma

Consider **(1,2) random walk** Y. For $1 \le m < k < n$, we have

$$Q_k^n(m,n) = \sum_{s=m+1}^{k} \mathbf{e}_1 A_s \cdots A_{n-1} \left(\frac{1 + \sum_{s=m+1}^{n-1} \mathbf{e}_1 A_s \cdots A_{n-1} \mathbf{e}_2^t}{1 + \sum_{s=m+1}^{n-1} \mathbf{e}_1 A_s \cdots A_{n-1} \mathbf{e}_1^t} \mathbf{e}_1^t - \mathbf{e}_2^t \right),$$

$$Q_k^{n+1}(m,n) = \sum_{s=m+1}^{k} \mathbf{e}_1 A_s \cdots A_{n-1} \left(\mathbf{e}_2^t - \frac{\sum_{s=m+1}^{n-1} \mathbf{e}_1 A_s \cdots A_{n-1} \mathbf{e}_2^t}{1 + \sum_{s=m+1}^{n-1} \mathbf{e}_1 A_s \cdots A_{n-1} \mathbf{e}_1^t} \mathbf{e}_1^t \right),$$

$$Q_k(m,n,+) = \frac{\sum_{s=m+1}^{k} \mathbf{e}_1 A_s \cdots A_{n-1} \mathbf{e}_1^t}{1 + \sum_{s=m+1}^{n-1} \mathbf{e}_1 A_s \cdots A_{n-1} \mathbf{e}_1^t}.$$

Lemma

Consider **(2,1) random walk** X. For $1 \le m < k < n$, we have

$$P_k(m,n,-) = \frac{\sum_{s=k}^{n-1} \mathbf{e}_1 A_s \cdots A_{m+1} \mathbf{e}_1^t}{1 + \sum_{s=m+1}^{n-1} \mathbf{e}_1 A_s \cdots A_{m+1} \mathbf{e}_1^t}.$$

Theorem 1

Suppose that $\lim_{k\to\infty} M_k = M$, $a+\theta \neq 0$, $b \neq 0$ and $bd \neq a\theta$. Then $\exists k_0 > 0$ such that for $k \geq k_0$ and $i, j \in \{1, 2\}$, we have

$$\frac{\mathbf{e}_i M_{k+1} \cdots M_{k+n} \mathbf{e}_j^t}{\xi_{k+1}^{-1} \cdots \xi_{k+n}^{-1}} \to \psi(i,j,k),$$

uniformly in k as $n \to \infty$; furthermore, if we assume further $\varrho \geq 1$, then for $k \geq k_0$ and $i = 1$, $j \in \{1, 2\}$, with the above $\psi(i,j,k)$, we have

$$\frac{\sum_{s=1}^{n+1} \mathbf{e}_i M_{k+s} \cdots M_{k+n} \mathbf{e}_j^t}{\sum_{s=1}^{n+1} \xi_{k+s}^{-1} \cdots \xi_{k+n}^{-1}} \to \psi(i,j,k),$$

uniformly in k as $n \to \infty$.

Lemma

Suppose that $\lim_{k\to\infty} M_k = M$, $a+\theta \neq 0$, $b \neq 0$, and $bd \neq a\theta$. Then there exist $0 < c_5 < c_6 < \infty$ and $N_1, N_2 > 0$ such that for $n > N_1, k > N_2$ we have

$$c_5 < \frac{\mathbf{e}_1 M_k \cdots M_{k+n} \mathbf{e}_1^t}{\varrho(M_k) \cdots \varrho(M_{k+n})} < c_6.$$

As a consequence, from Theorem 1 and the above lemma, we get

$$\mathbf{e}_1 M_k \cdots M_{k+n} \mathbf{e}_1^t \sim \xi_{k+1}^{-1} \cdots \xi_{k+n}^{-1} \asymp \varrho(M_k) \cdots \varrho(M_{k+n}).$$

If one wants to show that

$$\xi_{k+1}^{-1} \cdots \xi_{k+n}^{-1} \sim \varrho(M_k) \cdots \varrho(M_{k+n}),$$

some stronger condition is required on the convergence manner of $M_k \to M$, see

Wang, H.-M., Sun, H.-Y.: Asymptotics of product of non-negative 2-by-2 matrices with applications to random walks with asymptotically zero drifts. *Linear Multilinear Algebra*, **DOI**: 10.1080/03081087.2021.2022083 (2022)

Wang, H.-M., Yao, H.: Two-type linear fractional branching processes in varying environments with asymptotically constant mean matrices. *J. Appl. Probab.* **59**(1), 224-255 (2022)

Idea of the proof of Theorem 2:

- ♢ The escape prob. of the random walks can be written as functions of product of matrices $A_k \cdots A_{k+n}$. So, using

$$\mathbf{e}_1 A_k \cdots A_{k+n} \mathbf{e}_1^t \sim c\xi_{k+1}^{-1} \cdots \xi_{k+n}^{-1}$$

  one can write the escape prob. as function of the product of continued fractions. Consequently, one can estimate and analyze accurately the escape prob.

- ♢ Using

$$\xi_{k+1}^{-1} \cdots \xi_{k+n}^{-1} \asymp \varrho(A_k) \cdots \varrho(A_{k+n}),$$

  one can transit the product of continued fractions to that of spectral radii of matrices.

The proof is very technical, see

Wang, H.-M., Tang L.-L.: Cutpoints of (1,2) and (2,1) random walks on the lattice of positive half line, *arXiv:* 2206.09402 (2022)

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